The Pontryagin Construction
& the Thm. of Hopf

$f : S^m \to S^k$

$$\begin{tikzcd} \overset{\cup}{M} \arrow[r, squiggly] & \pi^{m-k}(M) \end{tikzcd}$$

Let M or., $\partial M = \emptyset$, M cpt. $m = \dim M$

$$f : M \to \underset{\text{k-dim sphere}}{S^k}$$

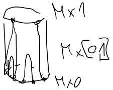
$N, N' \subseteq M$ cpt., boundaryless ~~of~~ submflds of

Def N is cobordant to N' within M if there ex. a cpt. submfd X of $M \times [0,1]$ s.t.

$$N \times [0, \varepsilon) \cup N' \times (1-\varepsilon, 1] \subseteq X \subseteq M \times [0,1]$$

$$\partial X = N \times 0 \cup N' \times 1 = X \cap \partial(M \times [0,1])$$

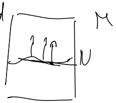
Rmk cob is an equiv. rel.



Def A framing of a submfd. $N \subseteq M$ is a smooth frame

$$v : N \to \underbrace{GL(TN^\perp)}_{(v_1, \dots, v_k)\ v_1, \dots, v_k \text{ is a basis of } TN^\perp} \qquad TN^\perp \subseteq TM$$

Rmk (N, v) is called a framed submfd



$(N, v), (N', v')$ are framed cobordant if there ex. a framed cobordism (X, u) s.t.

$$u|_{N \times 0} = v \quad u|_{N \times 1} = w$$

This is an equiv. rel.

Pontryagin ~~mfd~~ constr.

$f: M \to S^k$ smooth

& $y \in S^k$ a regular value of f

choose a pos. or. basis ν of $T_y S^k$

set of crit val. of f is a nowhere dense set

Construct a framed submfd. of M as follows:

$f^{-1}(y) \subseteq M$ is a smooth submfd of codim k

$\forall x \in f^{-1}(y)$ $d_x f$ maps $T_x N^\perp$ isomorphically onto $T_y S^k$

$\Rightarrow \exists!\, w_i \in T_x N^\perp$ s.t. $d_x f(w_i) = \nu_i$

$\Rightarrow w = f^* \nu$ is a framing of $f^{-1}(y)$

Def $(f^{-1}(y), f^* \nu)$ the Pontryagin mfd. ass. to f

Thm A If y' is another reg. value of f & ν' is another pos. or. basis of $T_{y'} S^k$

$\Rightarrow (f^{-1}(y'), f^* \nu')$ is framed cob. to $(f^{-1}(y), f^* \nu)$

$f: M \to S^k \rightsquigarrow [(f^{-1}(y), f^* \nu)]_{cob}$

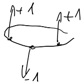
g

Thm B $f, g: M \to S^k$ are smoothly homotopic if and only if the ass. Pontryagin mfds are framed cobordant

$[f: M \to S^k]_{hom} \xrightarrow[\text{inj.}]{} [(f^{-1}(y), f^* \nu)]_{cob}$

Thm C Every framed submfd of codim k is a Pontryagin mfd of a map $f: M \to S^k$

Lemma 1 If v, v' are two pos. or. bases at y



$\Rightarrow (f^{-1}(y), f^{*}v)$ is framed cob. to $(f^{-1}(y), f^{*}v')$

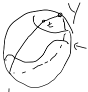
Proof Choose a path betw v and v' in $GL^{+}(T_y S^k)$ ⇝ gives rise to a framed cob.

Lemma 2 If y is a reg. of f and z close enough to y

$\Rightarrow f^{-1}(z)$ is framed cob. to $f^{-1}(y)$

Proof

∃ nbh. U of y (convex)



← crit val. of f

cont. only reg. values

Let $r_t : S^k \to S^k$ be a one-par. fam. of rot.

s.t. $r_1(y) = z$

$r_t = id$ near 0

$r_t = r_1$ near 1

$r_t^{-1}(z)$ lies on the arc betw. y and z

$\Rightarrow r_t^{-1}(z)$ reg. val of ∀ f

Def: $F(x,t) = r_t \circ f(x)$

$\Rightarrow \forall t \in [0,1]$ z is a reg. val. of $r_t \circ f$

$\Rightarrow F$

$\Rightarrow F^{-1}(z)$ is a framed cob. betw. $f^{-1}(z)$ & $(r_1 \circ f)^{-1}(z) = f^{-1}(y)$

Lemma 3 If f & g are smoothly homotopic

y as reg. val. of both

$\Rightarrow f^{-1}(y)$ is framed cob. to $g^{-1}(y)$

Proof Let F betw. f & g

choose a reg. z of F close enough to y s.t.

- $f^{-1}(y)$ fr. cob. to $f^{-1}(z)$
- $g^{-1}(y)$ " " " $g^{-1}(z)$

$\Rightarrow$ $F^{-1}(z)$ is ~~&~~ a fr. cob. betw $f^{-1}(z)$ & $g^{-1}(z)$

□

Proof of Thm A

Given reg. val. y, z of f, choose rot. $r_t : S^k \to S^k$ s.t. $r_0 = id$ $r_1(y)$ ~~z~~ $= z$

$\Rightarrow$ $f \simeq r_1 \circ f$

$\Rightarrow$ $f^{-1}(z)$ fr. cob. to $(r_1 \circ f)^{-1}(z)$

$= f^{-1}(r_1^{-1}(y))$

$= f^{-1}(y)$

□

$N \subseteq M$ fr. submfd., $\partial N = \emptyset$ of codim k

Tubular nbh. Thm.

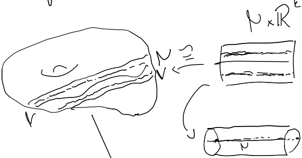
There ex. a nbh. V of N diffeomorphic to $N \times \mathbb{R}^k$, and the frame ν corresponds to the std. basis of $\mathbb{R}^k$

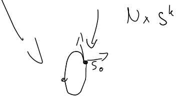
$N \times 0 \cong N$

Proof of Thm C

(N, ν) Choose a tub. nbh V

$g: N \times \mathbb{R}^k \to V \subseteq M$





Let $f,g: M \to S^k$ be smooth maps with reg. val. y

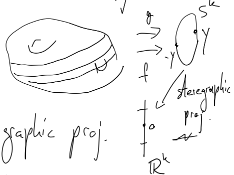
Lemma 4 $(f^{-1}(y), f^*\nu)$ is equal to $(g^{-1}(y), g^*\nu)$

$\Rightarrow f \simeq g$

Proof Set $N = f^{-1}(y) = g^{-1}(y)$

$f^*\nu = g^*\nu$ means $\forall x \in N : d_x f = d_x g$

First ass. that $f = g$ in a nbh. V of N



$h: S^k \setminus y \to \mathbb{R}^k$ stereographic proj.

Then define $F(x,1) = f(x)$ $x \in V$

$F(x,t) = h^{-1}(t\,h(f(x)) + (1-t)\,h(g(x)))$

$x \notin V$

does the job

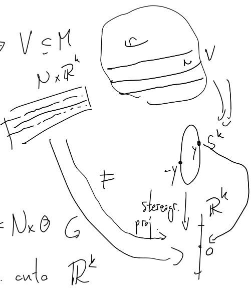
- Need to deform f into g on a nbh. of N without mapping new points into y

Choose a tub. nbh.

$N \times \mathbb{R}^k \to V \subseteq M$

small enough s.t.

$-y \notin f(V), g(V)$



$\leadsto F, G : N \times \mathbb{R}^k \to \mathbb{R}^k$

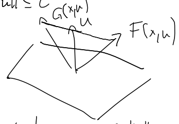
with $F^{-1}(0) = G^{-1}(0) = N \times 0$

$d_{(x,0)}F = dG_{(x,0)} =$ proj. onto $\mathbb{R}^k$

Can find $c > 0$ s.t.

$F(x,u)\cdot u > 0$ for $0<\|u\|\le c$

$G(x,u)\cdot u > 0$



$\Rightarrow (1-t)F(x,u) + t\,G(x,u) \neq 0$

won't map any new points into 0 if $\|u\|\le c$

Choose a sm map $\lambda: \mathbb{R}^k \to \mathbb{R}$ with

$\lambda(u) = 1$ $\|u\| \le c/2$

$\lambda(u) = 0$ $\|u\| \ge c$

$\Rightarrow F_t(x,u) = (1-\lambda(u)t)F(x,u) + \lambda(u)t\,G(x,u)$

$\Rightarrow$ deforms F into a map F_1 which is equal to G for $\|u\| \le c/2$ and equal to F for $\|u\| \ge c$

$\rightarrow$ ext. this to M

$\Rightarrow$ $f \underset{N.}{\simeq} f_1$ ← coincides with g on a nbh. of

□

Proof of Thm B

$f \simeq g$ smoothly hom. $\xrightarrow{\text{Lemma 3}}$ $f^{-1}(y)$ fr. cob. to $g^{-1}(y)$

Conv. given a framed cob. (X,w) betw. $f^{-1}(y)$ & $g^{-1}(y)$

$\underset{\text{in Thm C}}{\overset{\text{Same as}}{\Rightarrow}}$ $\exists$ a map $F: M\times[0,1] \to S^k$ s.t.

$(F^{-1}(y), F^{*}v) = (X,w)$

$\Rightarrow$ F_0 and f have the same Pontryagin mfd

F_1 " g " " " " "

$\overset{\text{Lemma 4}}{\Rightarrow}$ $F_0 \simeq f$, $F_1 \simeq g$ $\Rightarrow$ $f \simeq g$ □

Rmk Thm A, B, C can be ext. to mfds with boundary

hom. classes of maps $(\partial M, \partial M) \to (S^k, s_0)$

$\longleftrightarrow$ cob. classes of submfds $N \subseteq \mathring{M}$ of codim k

If $k > \frac{1}{2}m + 1 \rightsquigarrow$ abelian group structure on $\pi^k(M, \partial M)$

M conn & oriented & $m = k$

$f: M \to S^m$

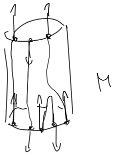
$\to$ framed submfd of dim 0

$\rightsquigarrow$ collection of finite pts with a pos. or neg. oriented basis at each of these points

$\operatorname{sgn}(x) = \pm 1$ dep. on whether the basis is pos. or. neg. or.

$\Rightarrow$ cobordism classes are determined $\sum_{x \in N} \operatorname{sgn}(x)$

We also have that $\deg f = \sum_{x \in f^{-1}(y)} \operatorname{sgn}(x)$

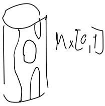


Thm of Hopf:

If M is conn. & $\partial M = \emptyset$ (cpt.)

$\Rightarrow f, g: M \to S^m$ are smoothly hom. iff they have the same degree

$\Rightarrow \pi_m(S^m) \cong \mathbb{Z}$



Thm: If M is conn & unoriented then $f, g: M \to S^{k\,m}$ are hom. iff they have the same degree mod 2

$S^m \to S^k$

$m = k + 1 \geq 4 \Rightarrow \exists$ two hom. classes

$\Rightarrow \pi_{k+1}(S^k) = \mathbb{Z}_2 \quad k \geq 3$

$m = k + 2 \geq 4 \Rightarrow \exists$ two hom. classes

$\Rightarrow \pi_{k+2}(S^k) = \mathbb{Z}_2 \quad k \geq 2$

Doesn't work for $m - k \geq 2$ anymore